Recitation 2. March 2

Focus: ***LU* and *LDU* factorizations, taking inverses, symmetric matrices, column spaces**

The *LU* factorization of a matrix A is the unique way of writing it:

$$\boxed{A = LU}$$

where L is a lower triangular matrix with 1's on the diagonal and U is in row echelon form. If A is square, then U is also square, in which case "row echelon form" means the same thing as "upper triangular". You can also write:

$$\boxed{A = LDU}$$

where both L and U have 1's on the diagonal, and D is diagonal. The discussion above works for almost all matrices A, and for those where it doesn't work, you can always write:

$$\boxed{PA = LDU}$$

for a suitable permutation matrix P.

The inverse of a square matrix A is the unique square matrix A with the property that $AA^{-1} = A^{-1}A = I$. One way to compute the inverse is to do Gauss-Jordan elimination on the augmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.

A symmetric matrix is one which is equal to its own **transpose**, i.e. its reflection across the diagonal.

The column space of a matrix is the vector space spanned by its columns.

1. Compute the $PA = LDU$ factorization of the matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

Solution: There are two choices for the 2×2 permutation matrix P:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The first one will not work, since $IA = A$ does not have an LU factorization (this is because Gaussian elimination will not work on the matrix A without a row exchange, due to the top pivot being right of the bottom pivot). Therefore, let us exchange the rows of A, which is achieved by multiplying with the second permutation matrix above:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

This matrix is already in row echelon form, so we don't need any row exchanges. However, we do need to multiply with the diagonal matrix:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

in order to get all the pivots to equal 1. We conclude that:

$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

Then just move the diagonal matrix to the right, but left multiplication with its inverse $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$$

We conclude $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $U = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$.

2. Compute the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 6 & -1 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

by Gauss-Jordan elimination on the augmented matrix $\left[\begin{array}{c|c} A & I \end{array}\right]$.

Solution: The augmented matrix is:

$$\begin{bmatrix} \boxed{1} & 6 & -1 & 1 & 0 & 0 \\ \boxed{3} & 1 & 2 & 0 & 1 & 0 \\ \boxed{2} & 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(pivots are boxed) The first step in Gauss-Jordan elimination is to subtract 3 times the first row from the second row and 2 times the first row from the third row:

$$\begin{bmatrix} \boxed{1} & 6 & -1 & 1 & 0 & 0 \\ 0 & \boxed{-17} & 5 & -3 & 1 & 0 \\ 0 & \boxed{-10} & 3 & -2 & 0 & 1 \end{bmatrix}$$

Then we subtract $\frac{10}{17}$ times the second row from the third row:

$$\begin{bmatrix} \boxed{1} & 6 & -1 & 1 & 0 & 0 \\ 0 & \boxed{-17} & 5 & -3 & 1 & 0 \\ 0 & 0 & \boxed{\dfrac{1}{17}} & -\frac{4}{17} & -\frac{10}{17} & 1 \end{bmatrix}$$

The next step is to make all pivots 1, by dividing the second row by -17 and multiplying the third row by 17:

$$\begin{bmatrix} \boxed{1} & 6 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & -\frac{5}{17} & \frac{3}{17} & -\frac{1}{17} & 0 \\ 0 & 0 & \boxed{1} & -4 & -10 & 17 \end{bmatrix}$$

To complete Gauss-Jordan elimination, we need to make the entries above the pivots 0. To do so, we first add $\frac{5}{17}$ times the third row to the second row:

$$\begin{bmatrix} \boxed{1} & 6 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & -1 & -3 & 5 \\ 0 & 0 & \boxed{1} & -4 & -10 & 17 \end{bmatrix}$$

Then we add -6 times the second row to the first row and 1 times the third row to the first row:

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 3 & 8 & -13 \\ 0 & \boxed{1} & 0 & -1 & -3 & 5 \\ 0 & 0 & \boxed{1} & -4 & -10 & 17 \end{bmatrix}$$

Thus, the inverse is:

$$A^{-1} = \begin{bmatrix} 3 & 8 & -13 \\ -1 & -3 & 5 \\ -4 & -10 & 17 \end{bmatrix}$$

3. Show that for any matrix A, the square matrix $S = A^T A$ is symmetric. For any vector $\boldsymbol{v}$, show that:

$$\boldsymbol{v}^T S \boldsymbol{v} \tag{1}$$

is a (1×1 matrix whose only entry is a) non-negative number.

Solution: S being symmetric boils down to the fact that $S^T = (A^TA)^T = A^T(A^T)^T = A^TA = S$. As for (1), if we consider the vector:

$$A\boldsymbol{v} = \boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

then:

$$\boldsymbol{v}^T S\boldsymbol{v} = \boldsymbol{v}^T A^T A\boldsymbol{v} = (\boldsymbol{v}^T A^T)(A\boldsymbol{v}) = (A\boldsymbol{v})^T(A\boldsymbol{v}) = \boldsymbol{w}^T\boldsymbol{w} = w_1^2 + ... + w_m^2 \geq 0$$

This non-negativity will play an important role in a few weeks' time.

4. Find numbers a, b such that the column space of the matrix:

$$A = \begin{bmatrix} 1 & a \\ b & 3 \\ 2 & 1 \end{bmatrix}$$

is the plane in xyz space determined by the equation $2x + y - 3z = 0$.

Solution: The column space in question consists of all linear combinations of the columns of the matrix A, i.e.:

$$\boldsymbol{v} = \lambda \begin{bmatrix} 1 \\ b \\ 2 \end{bmatrix} + \mu \begin{bmatrix} a \\ 3 \\ 1 \end{bmatrix}$$

for all numbers λ and μ. Such vectors $\boldsymbol{v}$ will lie in the plane in question if and only if the individual columns:

$$\begin{bmatrix} 1 \\ b \\ 2 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a \\ 3 \\ 1 \end{bmatrix}$$

lie in the plane in question. And for this to happen, the coordinates of the two columns must satisfy the equation $2x + y - 3z = 0$, i.e.:

$$2 \cdot 1 + b - 3 \cdot 2 = 0 \qquad \text{and} \qquad 2 \cdot a + 3 - 3 \cdot 1 = 0$$

By solving the equations above, we see that we need $a = 0$ and $b = 4$.